ICON Journal of Engineering Applications of Artificial Intelligence
Volume 01 | Issue 02 | 2025
e-ISSN: 3051-3820
p-ISSN: 3051-3812
Journal homepage: https://iconpublishers.com/icon-j-eng-app-ai/

DOI: 10.5281/zenodo.16791498

Research Article

A Hybrid Block Technique with Two-Step Optimization for Handling General Third Order Ordinary Differential Equations

*** Joshua, Solomon**
Department of Mathematical Sciences, Taraba State University, Jalingo

Corresponding author: Joshua, Solomon
Department of Mathematical Sciences, Taraba State University, Jalingo

Received Date: 02 July 2025
Published Date: 11 Aug. 2025

Abstract
In this research article, a hybrid technique with two-step optimization for handling general third-order ordinary differential equations is derived and implemented. The method was designed with the use of interpolation and a collocation approach, using power series as the basis function. We investigate the method's properties, such as order, convergence, consistency, zero-stability and region of absolute stability. The methods were tested on some third order ODEs problems and the outcome from the numerical examples showed that the new methods in the study performed better than those compared with in the literature.

Keywords: Two-step, Third Derivative, Optimized hybrid block, Local Truncation Error.

I. INTRODUCTION

Consider the general third order ordinary differential equations (ODEs) using power series of order seven given of the form

$$y(t) = \sum_{j=0}^{k} a_j t^j \qquad (1)$$

Which is recommend as general third order derivative solution of initial value problems of the form

$$y'''(t) = f(t, y, y', y''), y(t_0) = y_0, y'(t_0) = y'_0, y''(t_0) = y_0'' \qquad (2)$$

In particular, third order ordinary differential equations arise in many physical problems such as electromagnetic waves, thin film flow, and gravity-driven flows. The solution of (2) has been discussed by various researchers among them are: Dalatu *et al.* [1] developed a hybrid block method for solving third-order derivative with initial value problems of ordinary differential equations. Ghadimi [2] developed a multi-step predictor-corrector method for delay differential equations. Atabo *et al.* [3] developed a selected single step hybrid block formula for solving third-order ordinary differential equations with application in thin film flow. Saidu, [4] introduced three members of a one-step optimized third derivative hybrid block methods family for solving general second-order initial value problems. Raymond *et al*. [5] proposed an optimized half-step scheme third derivative method for testing higher order initial value problems. Soomro *et al.* [6] developed an optimized hybrid block Adam method for solving first order ordinary differential equations. Adam block method was design for the solution of linear and nonlinear first-order initial value problems in ordinary differential equations. Moses and Akintoye [7] developed an advanced two-step block method that integrates optimization techniques to enhance solution accuracy for third-order differential equations. Sabo *et al.* [8] developed the simulation of linear block algorithm for modelling third order highly stiff problem without reduction to a system of first order ordinary differential equation to address the weaknesses in reduction method Babatunde [9] proposed optimization strategies in hybrid methods, focusing on error minimization and enhanced convergence for higher-order ordinary differential equations. Sadiq and Ahmed [10] introduce a class of two-step hybrid block methods for solving third-order differential equations. The authors optimized the block method by incorporating error minimization strategies and improving computational efficiency.

The paper was organized as follows: the next section shows the methodological development of the optimized two-step method. The basic conditions of the method are analyzed; these are convergence and stability region, numerical experiments. The effectiveness of the scheme is confirmed on some samples and the result is discussed in Section 3. Section 4 is the conclusion.

2. Derivation of the Methods

To derive the pair of two-step third derivative methods, the finite power series function of the form

$$y(t) = \sum_{j=0}^{9} a_j\, t^j \tag{2}$$

is used as basis function.
By differentiating equation (2) thrice gives

$$y'''(t) = \sum_{j=0}^{9} j(j-1)(j-2)a_j t^{j-3} \tag{3}$$

Substituting (3) into (1) gives

$$y'''(t) = f(t,y,y',y'') = \lambda_0\theta_0 + h\lambda_1\theta'_n + h^2\lambda_2\theta''_n + \sum_{j=0}^{9} j(j-1)(j-2)a_j t^{j-3}$$

$$, i = 1.\ldots.,3 \tag{4}$$

Interpolating (2) and it first and second derivative at t_n and collocating (4) at all points $t_{n+\alpha} = t_n + \alpha h, \alpha = (0,r,v,1,s,u,2)$, gives

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & \frac{(r)}{1!} & \frac{(v)}{1!} & \frac{(1)}{1!} & \frac{(s)}{1!} & \frac{(u)}{1!} & \frac{(2)}{1!} \\ 0 & 0 & 0 & 0 & \frac{(r)^2}{2!} & \frac{(v)^2}{2!} & \frac{(1)^2}{2!} & \frac{(s)^2}{2!} & \frac{(u)^2}{2!} & \frac{(2)^2}{2!} \\ 0 & 0 & 0 & 0 & \frac{(r)^3}{3!} & \frac{(v)^3}{3!} & \frac{(1)^3}{3!} & \frac{(s)^2}{2!} & \frac{(u)^3}{3!} & \frac{(2)^2}{2!} \\ 0 & 0 & 0 & 0 & \frac{(r)^4}{4!} & \frac{(v)^4}{4!} & \frac{(1)^5}{4!} & \frac{(s)^3}{3!} & \frac{(u)^4}{4!} & \frac{(2)^3}{3!} \\ 0 & 0 & 0 & 0 & \frac{(r)^5}{5!} & \frac{(v)^5}{5!} & \frac{(1)^5}{5!} & \frac{(s)^5}{5!} & \frac{(u)^5}{5!} & \frac{(2)^5}{5!} \\ 0 & 0 & 0 & 0 & \frac{(r)^6}{6!} & \frac{(v)^6}{6!} & \frac{(1)^6}{6!} & \frac{(s)^6}{6!} & \frac{(u)^6}{6!} & \frac{(2)^6}{6!} \end{bmatrix} \begin{bmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \\ \lambda_7 \\ \lambda_8 \\ \lambda_9 \end{bmatrix} = \begin{bmatrix} \theta_n \\ \theta'_n \\ \theta''_n \\ \mu_n \\ \mu_{n+r} \\ \mu_{n+v} \\ \mu_{n+1} \\ \mu_{n+s} \\ \mu_{n+u} \\ \mu_{n+2} \end{bmatrix} \tag{5}$$

Equation (5) is solved for unknowns by applying the Crammer's method gives the coefficient $\lambda_j\left(j = 0(1)9\right)$ yield the continuous hybrid scheme which is given by

$$y'(t) = \lambda_0\theta_{n+i} + h\lambda_1\theta'_{n+i} + h^2\lambda_2\theta''_{n+i} + h^3\begin{bmatrix}\lambda_3\mu_n + \lambda_4\mu_{n+r} + \lambda_5\mu_{n+v} + \lambda_6\mu_{n+1} + \\ \lambda_7\mu_{n+s} + \lambda_8\mu_{n+u} + \lambda_9\mu_{n+2}\end{bmatrix} \tag{6}$$

The coefficient of θ_{n+i}, $i = 0$ and μ_{n+j}, $j = 0,r,v,1,s,u,2$ gives

$$\theta_{n+j} = \lambda_0\theta_n + h\lambda_1\theta'_n + h^2\lambda_2\theta''_n + h^3\begin{bmatrix}\lambda_3\mu_n + \lambda_4\mu_{n+r} + \lambda_5\mu_{n+v} + \lambda_6\mu_{n+1} + \\ \lambda_7\mu_{n+s} + \lambda_8\mu_{n+u} + \lambda_9\mu_{n+2}\end{bmatrix} \tag{7}$$

Where

$$\lambda_0 = 1, \qquad \lambda_1 = k. \qquad \lambda_2 = \frac{1}{2}k^2$$

$$\lambda_3 = \frac{k^2}{10080rsuv}\begin{bmatrix} 72k^4v - 15k^5v + 48k^4 - 45k^5 + 10k^6 + 168k^2rs - 12k6k^3rs + 24k^4rs + \\ 168k^2ru - 126k^3ru + 168k^2rv + 168k^2su + 24k^4ru - 126k^3uv + 24k^4uv \\ +252k^2rsu - 42k^3rsu + 252rsv - 42k^3rsv + 252k^2ruv - 42k^3ruv + 252k^2suv \\ -420krsu - 420krsv - 420kruv - 420ksuv + 1680rsuv + 84k^2rsuv - 630krsuv \end{bmatrix}$$

$$\lambda_4=\frac{k^4}{5040r(r-s)(r-u)(r-v)(r-1)(r-2)}\begin{bmatrix}72k^3s-84k^2s-15k^4s-84k^2u+72k^3u-84k^2v-15k^4u\\+72k^3v-15k^4v+10k^5-126k^2su+24k^3su-126k^2sv+\\24k^3sv-126k^2uv+24k^3uv+168ksu+168ksv+168kuv-\\420suv-42k^2suv+252ksuv\end{bmatrix}$$

$$\lambda_s=\frac{k^2}{5040v(r-v)(s-v)(u-v)(v-1)(v-2)}\begin{bmatrix}72k^3r-84k^2r-84k^2s-15k^4r+72k^3s-15k^4s-84k^2u\\+72k^3u-15k^4u-45k^4+10k^5-126k^2rs+24k^3rs-126\\k^2ru+24k^3ru-126k^2su+24k^3su+168krs+168kru+\\168ksu-420rsu-42k^2rsu+252krsu\end{bmatrix}$$

$$\lambda_6=\frac{k^4}{5040(r-1)(s-1)(u-1)(v-1)}\begin{bmatrix}48k^3r-15k^4r+48k^3s-15k^4s+48k^3u-15k^4u+48k^3v-15k^4v\\-30k^4+10k^5-84k^3uv-42k^2rsu-42k^2rsv-42k^2ruv-42k^2suv\\+168krsu+168krsv+168kruv+168ksuv-420rsuv+84krsuv\end{bmatrix}$$

$$\lambda_7=-\frac{k^4}{5040s(r-s)(s-u)(s-1)(s-2)}\begin{bmatrix}72k^3r-84k^2r-15k^4r-84k^2u+72k^3u-84k^2v-15k^4u\\+72k^3v-15k^4v+48k^3-45k^4+10k^5-126k^2ru+24k^3ru\\-126k^2rv+24k^3rv-126k^2uv+24k^3uv+168kru+168krv\\+168kuv-420ruv--42k^2ruv+252kruv\end{bmatrix}$$

$$\lambda_8=\frac{k^4}{5040u(r-u)(s-u)(u-v)(u-1)(u-2)}\begin{bmatrix}72k^3r-84k^2r-84k^2s-15k^4r+72k^3s-15k^4s-84k^2v\\+72k^3v-15k^4v+48k^3-45k^4+10k^5-126k^2rs\\+24k^3rs-126k^2rv+24k^3rv-126k^2sv+24k^3sv\\+168krs+168krv+168ksv-420rsv-42k^2rsv+252krsv\end{bmatrix}$$

$$\lambda_9=\frac{k^4}{10080(r-2)(s-2)(u-2)(v-2)}\begin{bmatrix}24k^3r-15k^4r+24k^3s-15k^4s+24k^3u-15k^4u+24k^3v\\-15k^4v-15k^4+10k^5-42k^2rs+24k^3sv-42k^2uv\\+24k^3uv-42k^2rsu-42k^2rsv-42k^2ruv-42k^2suv\\+84krsu+84krsv-84kruv+84ksuv-210rsuv+84krsuv\end{bmatrix}$$

(8)

The first and second derivative of equation (6) is given by

$$hy'(t_n+zh)=\lambda_0\theta'_0+h\lambda_1\theta''_n+h^2\lambda_2\theta'''_n+h^3\begin{bmatrix}\lambda_3\mu_n+\lambda_4\mu_{n+r}+\lambda_5\mu_{n+v}+\lambda_6\mu_{n+1}+\\\lambda_7\mu_{n+s}+\lambda_8\mu_{n+u}+\lambda_9\mu_{n+2}\end{bmatrix} \quad (9)$$

$$h^2y''(t_n+zh)=h^2\lambda_2\theta''_0+h^3\begin{bmatrix}\lambda_3\mu_n+\lambda_4\mu_{n+r}+\lambda_5\mu_{n+v}+\lambda_6\mu_{n+1}+\\\lambda_7\mu_{n+s}+\lambda_8\mu_{n+u}+\lambda_9\mu_{n+2}\end{bmatrix} \quad (10)$$

By putting k=1 in (8) we get a multistep formula to approximate the solution of (1) at the point t_{n+1} that is given by

$$\theta_{n+1}=\theta_n+h\theta'_n+\frac{1}{2}h^2\theta''_n+\left[\begin{array}{l} \dfrac{\begin{array}{l}-27r-27s-27u-27v+66rs+66ru+66rv+66su+66sv+66uv\\-219rsu-210rsv-210ruv-210suv+1134rsuv+13\end{array}}{10080rsuv}\mu_n \\ -\dfrac{27s+27u-27v-66su-66sv-66uv+210suv-13}{5040r(r-1)(r-2)(r-v)(r-u)(r-s)}\mu_{n+r}+\dfrac{27r+27s+27u-66rs-66ru-66su+210rsu-13}{5040v(r-1)(r-2)(u-v)v(r-v)}\mu_{n+v} \\ +\dfrac{\begin{array}{l}-33r-33s-33u-33v+60rs+60ru+60rv+60su+60sv\\+60sv+60uv-126rsu-126rsv-126ruv-126suv+336rsuv+20\end{array}}{5040(v-1)(u-1)(s-1)(r-1)}\mu_{n+1} \\ +\dfrac{27r+27u+27v-66ru-66rv-66uv+210ruv-13}{5040s(s-1)(s-2)(s-v)(s-u)(r-s)}\mu_{n+s}-\dfrac{27r+27s+27v-66rs-66rv-66sv+210rsv-13}{5040u(u-1)(u-2)(u-v)(s-u)(r-u)}\mu_{n+u} \\ -\dfrac{\begin{array}{l}-9r-9s-9u-9v+18rs+18ru+18ru+18su+18sv+18uv\\-42rsu-42rsv-42ruv-42suv+126rsuv+5\end{array}}{10080(v-2)(u-2)(s-2)(r-2)}\mu_{n+2} \end{array}\right] \quad (11)$$

Expanding equation (11) using Taylor series around the point t_n which we gives the corresponding local truncation error as

$$L(\theta(t_{n+1}):h)=\frac{1}{25401600}\begin{pmatrix}-13r-13s-13u-13v+27rs+27rv+27sv+27uv-66rsu\\-66rsv-66ruv-66suv+210rsuv+7\end{pmatrix} \quad (12)$$

We optimized (12) to gives a new value of v by equating it to zero, keeping r, s, u as free parameter by assigning the values $r=\frac{1}{3}, s=\frac{4}{3}, u=\frac{5}{3}$ in (12) we obtain $v=\frac{101}{179}$. Substituting the values of v, s, u and r into equations (5) gives three equations, one for approximating the approximate solution and the other two for approximating the first and second derivative at all points which after employing Gauss elimination methods and evaluating the coefficient gives the general equations in block form

$$A^{(0)}w_m^{(i)}=\sum_{i=0}^{2}h^{(i)}e_i\theta^{(i)}+h^{(3-i)}\sum_{j=0}^{2}\lambda_j\left(\mu_n\left(\theta_{m+j}\right)+\mu_n\left(\theta_{n+j}\right)\right) \quad (13)$$

Where

$$w_m^{(i)}=\left[\theta_{n+r}^i\ \theta_{n+v}^i\ \theta_{n+1}^i\ \theta_{n+s}^i\ \theta_{n+u}^i\ \theta_{n+2}^i\right]^T \qquad w_m^{(i)}=\left[\theta_{n-r}^i\ \theta_{n-v}^i\ \theta_{n-1}^i\ \theta_{n-s}^i\ \theta_{n-u}^i\ \theta_{n}^i\right]^T$$

$$\mu_m^{(i)}=\left[\mu_{n+r}^i\ \mu_{n+v}^i\ \mu_{n+1}^i\ \mu_{n+s}^i\ \mu_{n+u}^i\ \mu_{n+2}^i\right]^T \qquad \mu_m^{(i)}=\left[\mu_{n-r}^i\ \mu_{n-v}^i\ \mu_{n-1}^i\ \mu_{n-s}^i\ \mu_{n-u}^i\ \mu_{n}^i\right]^T$$

and $A=6\times 6$ matrix

when i=0

$$e_0=\begin{bmatrix}0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\end{bmatrix},\quad e_1=\begin{bmatrix}0&0&0&0&0&\frac{1}{3}\\0&0&0&0&0&\frac{101}{179}\\0&0&0&0&0&1\\0&0&0&0&0&\frac{4}{3}\\0&0&0&0&0&\frac{5}{3}\\0&0&0&0&0&2\end{bmatrix},\quad e_2=\begin{bmatrix}0&0&0&0&0&\frac{1}{18}\\0&0&0&0&0&\frac{10201}{64082}\\0&0&0&0&0&\frac{1}{2}\\0&0&0&0&0&\frac{8}{9}\\0&0&0&0&0&\frac{25}{18}\\0&0&0&0&0&2\end{bmatrix},$$

$$\mu_0=\begin{bmatrix}0&0&0&0&0&\frac{1403251}{13744080}\\0&0&0&0&0&\frac{5514925007325007}{55262110587467280}\\0&0&0&0&0&\frac{17651}{169680}\\0&0&0&0&0&\frac{88066}{859005}\\0&0&0&0&0&\frac{287255}{2748816}\\0&0&0&0&0&\frac{2087}{21210}\end{bmatrix},$$

$$\mu_n = \begin{bmatrix} \frac{33689}{83328} & -\frac{509431134921491527}{2088222893185927680} & \frac{1342199}{10614240} & -\frac{160883}{2081520} & \frac{150809}{5967360} & -\frac{61877}{17486280} \\ \frac{2318190934691760327}{4568334475230628480} & -\frac{1515110730126587}{15230081810711040} & \frac{492553533033486487}{4789382917580497560} & -\frac{493018889570134659}{7607750557541328880} & \frac{468385400869420617}{21810112978520419840} & -\frac{21515259974137661}{7101181210489545480} \\ \frac{62703}{138880} & \frac{1414446878131603}{8593509848501760} & \frac{5503}{14560} & -\frac{30411}{231280} & \frac{25569}{663040} & -\frac{1109}{215880} \\ \frac{304}{651} & \frac{526305815118736}{4078560338253765} & \frac{190048}{331695} & \frac{5638}{130095} & \frac{256}{11655} & -\frac{7576}{2185785} \\ \frac{112075}{249984} & \frac{698999910704571 25}{417644578637185536} & \frac{1052875}{2122848} & \frac{123625}{416304} & \frac{193285}{1193472} & -\frac{29425}{3497256} \\ \frac{873}{1736} & \frac{32894113444921}{537094365531360} & \frac{1823}{2730} & \frac{1611}{28910} & \frac{21411}{41440} & \frac{2626}{26985} \end{bmatrix}$$

3. Analysis and implementation

In this subsection, analysis of basic properties of the newly derived methods shall be carried out. These properties include order and error constant, local truncation error, consistence, zero-stability, convergence, stability polynomial and region of absolute stability.

3.1 Orders and Error Constants

Consider the linear operator L associated with the implicit hybrid block methods defined as

$$L[\theta(t_n : h)] = \sum_{j=0}^{k} \left[\alpha_j \theta(t_n + jh) - h^3 \lambda_j \theta'''(t_n + jh)\right]$$

Where $y(t_n)$ is an arbitrary test function that is continuous and differentiable in the interval [a, b]. Obtaining the Taylor series expansions of $\theta(t_n + jh)$ and $\theta'''(t_n + jh)$ about t_n and collecting the coefficient of h^p gives;

$$L[\theta(t_n : h)] = c_1\theta(t_n) + c_1 h\theta'(t_n) + c_2 h^2 \theta''(t_n) + \ldots\ldots + c_p h^p \theta^{(p)}(t_n) + \ldots\ldots$$

Where $c_j's$ for j = 0, 1, 2, 3…. [11]

From (13), the linear multistep method has order p if

$$l[y(x) : h] = 0(h^{p+1}), c_0 = c_1 = c_2 = \ldots = c_{p+2} = 0, c_{p+3} \neq 0$$

$$\left[\frac{5274641}{163493359545600}, \frac{43664}{212881978575}, \frac{13}{25401600}, \frac{4303472559874195697909}{60134825116519125131428416 00}, \frac{3440875}{2179911460608}, \frac{236}{97339725}, \frac{7932311}{27248893257600}, \frac{307543}{425763957150}, \frac{14059}{12459484800}, \frac{332595749268839028277}{248263039471867947790300800}, \frac{2417875}{1089955730304}, \frac{527}{194679450}, \frac{900911}{605530961280}, \frac{21457}{18922842540}, \frac{3221}{2491896960}, \frac{5766240721685330 33}{455528512792418252826240}, \frac{295075}{121106192256}, \frac{767}{15689721600}\right]^T$$

3.2 Zero Stability of the Block Method

The new block method is zero stable if the first characteristic polynomial $\rho(w) = \det\left[\sum_{j=0}^{k} A^{(i)} w^{k-i}\right] = 0$ and satisfies

$$|w_j| \leq 1$$

$$
w\begin{bmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&&0&0&0&0&0&1
\end{bmatrix}
-
\begin{bmatrix}
1&0&0&0&0&\frac{1}{3}&0&0&0&0&\frac{1}{18}&0&0&0&0&0&0&0\\
1&0&0&0&0&\frac{101}{179}&0&0&0&0&\frac{10201}{64082}&0&0&0&0&0&0&0\\
1&0&0&0&0&1&0&0&0&0&\frac{1}{2}&0&0&0&0&0&0&0\\
1&0&0&0&0&\frac{4}{3}&0&0&0&0&\frac{8}{9}&0&0&0&0&0&0&0\\
1&0&0&0&0&\frac{5}{3}&0&0&0&0&\frac{25}{18}&0&0&0&0&0&0&0\\
1&0&0&0&0&2&0&0&0&0&2&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&\frac{1}{3}&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&\frac{101}{179}&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&\frac{4}{3}&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&\frac{5}{3}&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{bmatrix} = 0
$$

$$w^{18} - 3w^{17} + 3w^{16} - w^{15} = w^{15}(w-1)^3$$

Solving the characteristic equation gives

$$\rho(w) = w^{15}(w-1)^3 = 0, \quad w = 0,0,0,0,0,0,0,0.0,0,0,0,0,0,0,1,1,1$$

Therefore, $|w_i| = |\ 0,0,0,0,0,0,0,0,0.0,0,0,0,0,0,1,1,1| \le 1$ the method is zero-stable.

3.3. Consistency

The optimized scheme is consistent, [12] since it has order more than or equal to one.

3.4 Convergence of the method

The optimized scheme is said to be convergent if and only if it is consistent and zero stable, [13]. Since the method is satisfies the two conditions, then the method converges

3.5. Zero Stability of Our Method

Definition: A third derivative optimized scheme is said to be zero-stable, if the roots $w_i, i = r, \mathrm{v}, 1, s, u, 2$ of the first characteristic polynomial $\rho(w) = 0$ that is

$\rho(w) = \det\left[\sum_{j=0}^{k} A^{(i)} w^{k-i}\right] = 0$ satisfies $|w| \le 1$ and for those roots with $w_i = 1$ multiplicity must not exceed two. Hence, our method is zero-stable, [14].

3.5 Stability Polynomial

The stability polynomial of our method is given by

$$h^6\left(\frac{101}{608958}w^6 \text{✍} \frac{257}{1217916}w^5\right) \text{✍} h^5\left(\frac{15397}{6089580}w^6 \square \frac{4688}{1522395}w^5\right) \square h^4\left(\frac{88447}{4059720}w^6 \text{✍} \frac{102527}{4059720}w^5\right) \text{✍} h^3\left(\frac{12403}{101493}w^6 \square \frac{13745}{101493}w^5\right) \square$$
$$h^2\left(\frac{15140}{33831}w^6 \text{✍} \frac{16130}{33831}w^5\right) \text{✍} h\left(\frac{3704}{3759}w^6 \square \frac{3814}{3759}w^5\right) \square w^6 \text{✍} w^5$$

The absolute stability region of the method is plotted and shown in figure1

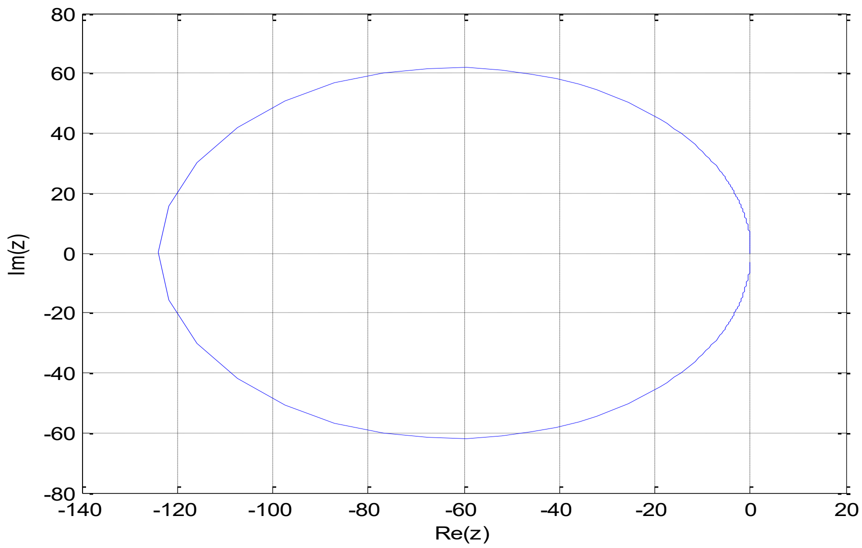

Figure1. The Region of Absolute Stability of the method

3.6 Numerical Examples

Problem 1

Consider the initial value problem below $y'''-y''+y'-y=0$, $y(0)=1,\ y'(0)=0,\ y''(0)=-1 : h=\frac{1}{100}$, Exact Solution $y(x)=\cos x$

Source: Tumba *et al.* (2021)

Example2. Consider the highly non-stiff third order ordinary differential equation

$$y'''(x)=3\cos(x),\quad y(0)=1,\ y'(0)=0,\ y''(0)=2 : h=\frac{1}{10}$$

The exact solution given by $y(x)=x^2-3\sin(x)+3x+1$

Source: Taparki *et al.* (2010)

Problem 3

Consider a highly stiff problem

$y'''+5y''+7y'+3y=0,\ y(0)=1,\ y'(0)=0,\ y''(0)=-1$, Exact Solution: $y(x)=e^{-x}+xe^{-x}$, $h=\frac{1}{10}$

Source: Tumba *et al.* (2021)

Table1. Showing the comparison of absolute error in our method with Raymond *et al.* [5] for problem two

x	Exact solution	Computed solution	Error in our method	Err in Raymond *et al* [5]
0.1	0.9999500004166652778	0.9999500004166652772	4.00e-20	2.55208e-12
0.2	0.9998000066665777781	0.9998000066665777788	3.00e-20	3.64210e-12
0.3	0.9995500337489875167	0.9995500337489875168	1.00e-20	4.5313e-12
0.4	0.9992001066609779401	0.9992001066609779405	6.00e-20	1.3406e-12
0.5	0.9987502603949662466	0.9987502603949662463	3.00e-20	3.28547e-12
0.6	0.9982005399352041655	0.9982005399352041655	6.00e-20	4.59125e-12
0.7	0.9975510002532795742	0.9975510002532795746	1.10e-19	5.47318e-12
0.8	0.9968017063026193848	0.9968017063026193845	1.31e-19	1.96524e-12
0.9	0.9959527330119942539	0.9959527330119942530	9.00e-19	2.34526e-12
1.0	0.9950041652780257660	0.9950041652780257654	1.60e-19	2.55587e-12

Table2. Showing the comparison of absolute error in our method with Taparki *et al.* [15] for problem three

x	Exact solution	Computed solution	Error in our method	Error in Taparki *et al.* [15]
0.1	1.0104997500595155431 0	1.01049975005951552960	1.350e-17	2.4800e-07
0.2	1.04399200761481635360	1.04399200761481628590	6.770e-17	7.3740e-06
0.3	1.10343938001598127470	1.10343938001598109520	1.795e-16	6.0542e-05
0.4	1.19174497307404852500	1.19174497307404811040	4.146e-16	2.5479e-04
0.5	1.31172338418739099920	1.31172338418739022780	7.714e-16	7.7602e-04
0.6	1.46607257981489392840	1.46607257981489259380	1.334e-16	1.9261e-03
0.7	1.65734693828692683900	1.65735020063250070330	3.262e-06	4.1505e-03
0.8	1.88793172730143171510	1.88794794834400722260	1.622e-05	8.3637e-03
0.9	2.16001927111754983460	2.15991439240417582420	1.048e-04	1.4774e-02
1.0	2.47558704557631048000	2.47511682136265474470	4.702e-04	2.4702e-02

Table3. Showing the comparison of absolute error in our method with Tumba *et al.* [16] for problem two

x	Exact solution	Computed solution	Error in our method	Err in Tumba *et al.* [16]
0.1	0.9953211598395555308	0.9953211598395558739	3.4311e-16	1.0434e-14
0.2	0.9824769036935782300	0.9824769036935797036	1.47296e-15	9.8731e-14
0.3	0.9630636868862332259	0.9630636868862359300	2.70461e-15	3.1317e-13
0.4	0.9384480644498950214	0.9384480644498990435	4.02221e-15	6.6668e-13
0.5	0.9097959895689501350	0.9097959895689551975	5.06255e-15	1.1507e-12
0.6	0.8780986177504422921	0.8780986177504482366	5.94415e-15	1.7445e-12
0.7	0.8441950164453961749	0.8441950164454026430	6.46881e-15	2.4220e-12
0.8	0.8087921354109988647	0.8087921354110056593	6.79516e-15	3.1554e-12
0.9	0.7724823535071383127	0.7724823535071451207	6.8083e-15	3.9178e-12
1.0	0.7357588823428846430	0.7357588823428913075	6.66385e-15	4.6852e-12

4. Conclusion

A hybrid technique with two-step optimization for handling general third-order ordinary differential equations was proposed using scientific workplace 5.5 versions for the derivation. The method is applied in block form and when analyzing the properties of the method, it was found to be zero-stable, consistent, and convergent. Also, the order and error constant are established.

It can be seen from the Table 1, 2, and 3 that our method performance better than the existing method of [14] and [15] when solving similar examples.

REFERENCES

1. Dalatu, P. I., Sabo, J., & Mathew, M. (2024). Numerical application of third derivative hybrid block methods on third order initial value problem of ordinary differential equations. *International Journal of Statistics and Applied Mathematics, 4*(6), 90–100.
2. Ghadimi, S. (2023). Multi-step predictor–corrector methods for delay differential equations. *Mathematical Methods in the Applied Sciences, 46*(5), 3784–3801. https://doi.org/10.1002/mma.8628
3. Atabo, V. O., Argawal, P., Kwala, A. K., & Anongo, N. R. (2020). Selected single-step hybrid block formula for solving third order ordinary differential equation. *FUDMA Journal of Science, 6*, 150–168. https://dx.doi.org/10.33003/fjs-2022-0606-1152
4. Saidu, D. Y. (2023). Family of one–step A-stable optimized third derivative hybrid block methods for solving general second-order initial value problems. *Al-Rafidain Journal of Computer Sciences and Mathematics, 17*(2), 125–140.
5. Raymond, D., Pantuvo, T. P., Lydia, A., Sabo, J., & Ajia, R. (2023). An optimized hoptimized half-step scheme third derivative methodg higher order initial value problems. *African Scientific Report, 2*(1), 76–85.
6. Soomro, H., Zainuddin, N., Daud, H., & Sunday, J. (2022). Optimized hybrid block Adams method for solving first-order differential equations. *Journal of Computers, Materials and Continua, 72*, 1–14. https://doi.org/10.32604/cmc.2022.025933
7. Moses, J., & Akintoye, A. (2022). Advanced two-step optimized block methods for third-order differential equations: Stability and convergence analysis. *Applied Numerical Mathematics, 168*, 123–139.
8. Sabo, J., Kyagya, T. Y., & Solomon, M. (2021). One-step hybrid block scheme for the numerical approximation for solution of third-order initial value problems. *Journal of Scientific Research and Reports, 27*(12), 51–61. https://doi.org/10.9734/jsrr/2021/v27i1230501
9. Babatunde, S. K. (2023). Optimized hybrid methods for higher-order differential equations. *Journal of Computational and Applied Mathematics, 426*, 175–190.

10. Sadiq, I., & Ahmad, R. (2018). A class of two–step hybrid block methods for solving third-order differential equations. *Numerical Algorithms, 77*(4), 1087–1107. https://doi.org/10.1007/s11075-017-0381-0
11. Lambert, J. D. (1991). *Numerical methods in ordinary differential equations*. John Wiley & Sons.
12. Abdulazeez, K. J. (2024). Proposed two-step hybrid block method for the numerical solution of third order differential equation. *International Journal of Mathematics and Statistics Invention, 12*(3), 29–39.
13. Henrici, P. (1962). *Discrete variable methods in ordinary differential equations*. Wiley.
14. Omole, E. O., & Ukpebor, L. K. (2020). A step-by-step guide on derivative and analysis of a new numerical method for solving fourth order ordinary differential equation. *Journal of Mathematics Letters, 6*(2), 13–31. https://doi.org/10.11648/j.jml.20200602.12
15. Taparki, R. M., Gurah, D., & Simon, S. (2010). An implicit Runge–Kutta method for solution of third order initial value problem in ODE. *International Journal of Numerical Mathematics, 6*, 174–189.
16. Tumba, P., Skwame, Y., & Raymond, D. (2021). Half-step implicit linear hybrid block approach of order four for solving third order ordinary differential equations. *Dutse Journal of Pure and Applied Sciences, 7*(2B), 124–133.